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DEFINITION BY ABSTRACTION AS A METHOD OF THE EMPIRICAL SCIENCES**

Abstract

In this paper, I analyze the structure of definition by abstraction employed in empirical sciences, whose specific feature is that it enables one to introduce a new magnitude on the basis of other, already known magnitudes. After reconstructing Aristotle's and Archimedes' treatment of the term "velocity," I characterize in general terms the importance of this method for empirical sciences and address the nature of this definition drawing on Peano's reconstruction. Next, I show that by means of that definition the magnitude *mass* can be introduced in classical mechanics, and the magnitude *value* in political economy drawing on the works of Ricardo. Then follows a critique of the nominalistic objections of Reichenbach and Dubislav against definition by abstraction. Finally, I show that this type of definition requires an in-depth semantic characterization, and this characterization should be based on the application of a hyperintensional semantic theory.

Keywords: definition by abstraction, Peano, Dubislav, magnitudes, hyperintensional semantics

The aim of this paper is to analyze the structure of a specific type of definition by abstraction — namely, one that is applied in empirical sciences with the purpose of defining new magnitudes on the basis of other, already known magnitudes.[1]

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[1] Following Karel Berka (1983), I employ in this paper, even if it deviates from standard English, expressions such as "magnitude *mass*" and "magnitude *velocity*" as referring to the very magnitude and not to its actual values, the latter being expressed by the term "sizes of magnitude *x*," for example, "sizes of magnitude *mass*," and "sizes of magnitude *velocity*."

I try to show that an analysis of this type of definition by abstraction as a method of the empirical sciences makes it possible to address important issues in the philosophy and methodology of science. Here I mean, first, the reconstruction of the method of theory construction by means of the introduction of new magnitudes and, second, the dispute between the realistic and nominalistic approaches in the philosophy and methodology of science.

I will start with Aristotle's and Archimedes' treatment of the term "velocity" and show how this treatment can be related to the method of definition by abstraction (MDA). Then, I shall deal with the structure of this method as reconstructed by Giuseppe Peano and characterize in general terms its importance for empirical sciences.

Next, I will address Isaac Newton's definition of magnitude *mass* in the *Principia* (1999) and, because of its circular nature, replace Newton's definition with one based on the magnitude *weight* understood in a pre-dynamical way, where this definition is by its nature a definition by abstraction. What will follow is an analysis of two opposing interpretations of the MDA: a nominalistic interpretation (which aims at a set-theoretical interpretation and, in fact, the elimination of the abstract entity introduced by this type of definition) and a realistic interpretation of that entity.[2]

I will attempt to show that these two possible interpretations are applicable to that definition of the magnitude *mass*. Their existence indicates a limitation of the definition of the magnitude *mass* on the basis of the predynamic magnitude *weight* and, in general, of the MDA as a method for introducing magnitudes in the empirical sciences.

To overcome that limitation, I employ an example from the empirical sciences — namely, the definition of the magnitude *labor* in the economic theory of David Ricardo, which shows that a realistic (i.e., non-nominalistic) definition of a new magnitude is possible and is one that is already a type of definition that lies outside the framework delineated by the MDA. On the basis of this example, it is also possible to show how that realistic definition transforms the MDA from its own *presupposition* into its own *outcome*, where by this transformation an extension of knowledge obtains.

Finally, by way of conclusion, I shall show that MDA that involves magnitudes requires an in-depth semantic characterization, where this characterization should be based on the application of a hyperintensional semantic theory.

[2] One could also say that nominalism eliminates abstract entities from the fundamental ontology, while realism claims that they are part of this ontology.

1. ARISTOTLE AND ARCHIMEDES ON VELOCITY

Aristotle declares the following in his *Physics*:

> The faster of two things traverses a greater magnitude in an equal time, an equal magnitude in less time, and a greater magnitude in less time, which is just the way faster is defined. (VI.2, 232a: 25-27; 2018: 104)

This statement can be symbolically restated as follows:[3]

(1) $t_A = t_B \rightarrow v_A : v_B :: s_A : s_B$

(2) $s_A = s_B \rightarrow v_A : v_B :: t_B : t_A$

For (1) and (2) it holds that they provide prescriptions concerning how to measure the ratio $v_A : v_B$ — namely, by means of either the ratio $s_A : s_B$ or the ratio $t_B : t_A$. According to the first prescription, the measurement has to be performed under the condition $t_A = t_B$, while according to the second prescription, the measurement has to be performed under the condition $s_A = s_B$. Expressions (1) and (2) can, at the same time, be viewed as definitions wherein their respective antecedent delineates the class of entities to which the definition can be applied: bodies that move in the same time or, alternatively, bodies that cover the same space.

While (1) and (2) enable one to measure the ratio of velocities, neither can be viewed as a definition of equality of the velocities of two bodies. Such a definition appears in Proposition 1 of Archimedes' *On Spirals*:

> If a certain point is carried along a certain line, moved at uniform speed with itself, and two lines are taken in it <=the original line>, the <lines> taken shall have to each other the very same ratio which the times <have to each other, =the times>, in which the point passed through. (Archimedes 2017: 36)[4]

Here the ratio of velocities is not measured as in (1) and (2), but the very equality of the velocities of two motions is defined. By also taking into account the definition-aspect of (1) and (2) it holds ("df" stands for definition):

(3) $(v_A = v_B) =_{df} s_A : s_B = t_A : t_B$

(4) $t_A = t_B \rightarrow v_A : v_B =_{df} s_A : s_B$

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[3] Here I draw on Oliver Schlaudt (2009), who shows that the content of Aristotle's statement can be expressed in this way. The symbol "*t*" refers to the magnitude *time*, "*v*" refers to the magnitude *velocity*, "*s*" refers to the magnitude *space*, "*A*" and "*B*" refer to moving bodies; ":" stands for ratio and "::" stands for "is as" — that is, for equality of ratios.

[4] The symbols < > surrounding insertions into the original are used by the translator of the Greek text.

(5) $s_A = s_B \rightarrow v_A : v_B =_{df} t_B : t_A$

Once statements $t_A = t_B$ and $s_A = s_B$ are understood as delineating the universes of discourse of entities for which definitions (4) and (5) are stated, they can be viewed as an exemplification of the MDA. It was first applied by Frege in § 64 of his *Grundlagen der Arithmetik*, but the priority of its detailed description belongs to Peano (1888).[5] Here I draw on Peano's explication of this method in (1894) and (1915).

2. PEANO ON DEFINITIONS BY ABSTRACTION

Peano states the following (1894, § 38: 45):

> There are concepts (*idées*) that are obtained by means of abstraction, and which constantly enrich mathematical sciences. . . . Let u be an object. By means of abstraction one deduces a new object φu. One cannot form an equality:
>
> φu = known expression,
>
> since φu is an object with a nature different from all those that have been until now considered. Hence one defines the equality and stipulates:
>
> $h_{u,v} . \supset : \varphi u = \varphi v . = . p_{u,v}$ Def.[6]
>
> where $h_{u,v}$ is the hypothesis about the objects u and v and $\varphi u = \varphi v$ is the equality being defined. It has the same meaning as $p_{u,v}$, which is a condition or a relation, between u and v, with a well-known meaning.

Peano then characterizes the condition/relation $p_{u,v}$ as being reflexive, symmetric, and transitive:

1. $\varphi u = \varphi u$; that is, $p_{u,u}$ has to be true for any u. Such a relation in which each object stands to itself is called *reflexive*.

2. $\varphi u = \varphi v \;.\supset.\; \varphi v = \varphi u$; that is, $p_{u,v} \supset p_{v,u}$. We call a relation *symmetric* when if u is in this relation to v, then v is in the same relation to u.

3. $\varphi u = \varphi v \;.\; \varphi v = \varphi w \;.\supset.\; \varphi u = \varphi w$; that is, $p_{u,v} \;.\; p_{v,w} . \supset . \; p_{u,w}$. Relations that satisfy this third condition are called *transitive*.

What Peano expressed in the definition (Def.) is the introduction of a new, previously unknown entity φ, where this introduction is based on the condi-

[5] For an analysis of the method of definition by abstraction in Peano (1888) and the contribution of the Peano school to the explication of this method, see Mancosu (2018).

[6] In Peano's notation dots stand for brackets while the symbol "$\supset$" can be replaced by "$\rightarrow$". Peano's Def. can then be rewritten in a more modern notation as $h_{u,v} \rightarrow ((\varphi u = \varphi v) \leftrightarrow p_{u,v})$.

tion/relation p jointly shared by/between u and v. So as the condition/relation $p_{u,v}$ is reflexive, symmetric, and transitive, for u and v the binary relation of equivalence holds. The abstraction that is at work in Def. stands for a filtering out of u and v of that entity in which they are identical: $\varphi u = \varphi v$.

This means that Peano's Def. in fact grounds the equivalence relation p between u and v in something, with respect to which they can be regarded as identical. In addition to its epistemological dimension — it enables to expand the existing stock of knowledge with a new concept — this grounding also has an *ontological* dimension. The real ground of the equivalence relation between u and v should be the existence of φ, with respect to which they are identical.[7] As I will show, it is precisely this ontological dimension of the method of definition of abstraction that is contested by nominalists.

The reason I deal with the MDA in connection with the empirical sciences should now be clear. Since this method allows one to introduce an *abstractum*, and thus a new object, the MDA is, according to Hermann Weyl, "a *creative* definition . . . through which new, ideal objects can be generated" (1949: 8).

As I will show now, the magnitude *mass* can be introduced in classical mechanics by a method that corresponds to that of MDA as expressed in Peano's Def.

3. HOW TO DEFINE THE MAGNITUDE *MASS* IN CLASSICAL MECHANICS

The magnitude *mass* appears for the first time in Newton's initial terminology as "quantity of matter" in the *Principia* (1999: 403):[8]

> DEFINITION 1. *Quantity of matter is a measure of matter that arises from its density and volume jointly.*

Newton then clarifies his terminology as follows: "I mean this quantity whenever I use the term 'body' or 'mass' in the following pages" (1999: 404).

The magnitude *density* thus should have, with respect to the above definition, the status of a magnitude already known prior to defining magnitude *mass*, but Newton does not define the magnitude *density* prior to Definition 1.[9]

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7 Paolo Mancosu refers to this ontological dimension of Peano's Def. as an "ontological spin given by Peano to definitions by abstraction" (2018: 265).

8 For a detailed analysis of the history of the magnitude *mass* see, for example, Jammer 1961.

9 Ernst Mach goes even further and characterizes Definition 1 as a "semblance of a

Newton, seemingly, indicates an alternative path of reasoning about the magnitude *mass* in the commentary on Definition 1, in which he declares that the magnitude mass "can always be known from a body's weight, for — by making very accurate experiments with pendulums — I have found it to be proportional to the weight" (1999: 404), and refers to his own experiments with pendulums which he describes in Book III of the *Principia*.

This description is based on the knowledge of the magnitude *weight*, which, however, is understood in a dynamical manner — namely, as a kind of force defined on the basis of magnitudes introduced by Newton in definitions that follow after Definition 1 and where all these definitions presuppose the knowledge of the magnitude *mass*. The proof of the existence of this circularity in the *Principia*, going from Definition 1 via Book II and Book III back to this definition, is relegated to the Appendix at the end of the paper.

Is it possible to introduce the magnitude *mass* while bypassing the magnitude *density* and at the same time eliminating the circular nature of Newton's reasoning?[10] In my view it is, but only when we realize that there are two reasons which together cause that reasoning to be circular.

The first reason is that, in the commentary on Definition 1, Newton relates the magnitude *mass* to magnitude *weight*. The second reason is that he simultaneously (by referring to the description of experiments with pendulums in Book III) understands the magnitude *weight* in a dynamical manner — namely, as a type of force. But once weight is understood in this way, as shown in that Appendix, Newton's reasoning turns circular.

One possible way to escape this circularity is by holding to Newton's grounding of magnitude *mass* in magnitude *weight* while simultaneously discarding the dynamical understanding of the latter magnitude as given in that grounding. We then face two different but still closely related, questions: (a) can magnitude *weight* be understood in a pre-dynamical manner, and if so, then (b) can magnitude *mass* be defined by means of such a pre-dynamical understanding of magnitude *weight*? It is possible, as I will now try to show, to formulate a positive answer to both of these questions.

Weight can be understood initially as a quality shared by bodies and one which we detect by manipulating them with our hands. We find out that they are heavy; they display a shared quality labeled "heaviness."[11] But how can we

definition (*Scheindefinition*). The concept of mass is not made clearer by representing mass as the product of volume and density, as density itself represents the mass of unit of volume" (1901: 255).

[10] For a more recent attempt to introduce the magnitude *mass*, see Martens (2017, 2018).

[11] The English noun "heaviness" coined from the adjective "heavy" sounds awkward; its German equivalent "Schwere" based on the adjective "schwer" does not.

then turn that quality into the magnitude *weight*? We can do that by comparing pairs of bodies x and y to find out if (i) x is greater than y in a certain respect, or (ii) x is equal to y in a certain respect, or finally, (iii) x is less than y in a certain respect.[12]

That comparison is practically performed on a double-pan balance by placing the bodies on the left (L) and right (R) pans and enables to determine which one of the three possible, mutually exclusive relations (i), (ii), and (iii) actually holds. The three possible actual states of affairs are as follows:

(1) Body x is said to be equal in weight to body y if and only if x on L balances y on R, and x on R balances y on L.

(2) Body x is said to be greater in weight than body y if and only if, with x on L and y on R, x descends, and with x on R and y on L, x descends.

(3) Body x is said to be less in weight than body y if and only if, with x on L and y on R, x ascends, and with x on R and y on L, x ascends.

In this way, the magnitude *weight* is not a dynamical magnitude based on the magnitude *mass* as given in the *Principia*, but a magnitude based on the experimental manipulation and experimental comparison of bodies. This means that the magnitude *weight* has, at this point, only the status of a pre-dynamical magnitude.

The introduction of the magnitude *mass* on the basis of the pre-dynamical magnitude *weight* can then be symbolically expressed as follows, employing Peano's Def.:

$$Bx = By \rightarrow (mx = my) =_{\mathrm{df}} (Wx =_{\mathrm{op}} Wy) \tag{6}$$

Here Bx and By stand for bodies x and y, respectively. $Bx = By$ corresponds to (in Peano's notation) hypothesis $h_{u,v}$ and picks out pairs from the universe of entities to which the definition could be applied. All such pairs should share the property of being a body. Wx and Wy stand for weight, understood in a pre-dynamical way, of x and y, respectively; mx and my stand for the mass of bodies x and y respectively, $mx = my$ corresponds to Peano's $\varphi u = \varphi v$. The sign "$=_{\mathrm{df}}$" stands for definition so that on its right side stands the *definiens*, while on its left side stands the *definiendum*. The direction of reasoning is thus from the right side to the left side; the magnitude *mass* is the introduced entity φ.

What one has to bear in mind is that in definition (6), its *definiens* states the identity of weights of two bodies as experimentally found out by the em-

[12] Here I found (Ellis 1960) helpful.

ployment of a beam scale in a *certain location* on Earth. Thus, the definition as a whole postulates the identity of the masses of these two bodies in the location where the beam scale indicated the identity of their weights. Stated otherwise, the definition excludes the possibility that two objects with different weights as detected on a scale in a certain location could have the same mass *in this location*.

Here, then, becomes apparent the difference between the pre-dynamical magnitude *weight* in definition (6) and the dynamical magnitude *weight* in the definition $W =_{\mathrm{df}} m \cdot g$ as given in classical mechanics. The latter definition states not only, as does (6), that two bodies with the same mass in the *same location* on Earth have the same weight, but it states, in addition, that two bodies with the same mass can have different weights in *different* locations on Earth.

Let me now delineate what "$=_{\mathrm{op}}$" employed in (6) stands for. It refers to the outcome of the comparison of bodies x and y on a double-pan balance as expressed above in (1). While "$=_{\mathrm{op}}$" in (6) corresponds to Peano's symbol "$p_{u,v}$" in Def., the relation to which the former refers, differs in one important aspect from that to which the latter refers. While both relations fulfill the conditions of symmetry and transitivity, "$=_{\mathrm{op}}$" refers to a relation that does not fulfill the condition of reflexivity. Using Peano's notation, $p_{u,u}$ does not hold for the introduction of magnitude mass by the MDA as stated in (6). The reason for this is that "$=_{\mathrm{op}}$" refers to an outcome of an operation of weighing performed on bodies x and y, and in such an operation neither of these bodies can be placed simultaneously on the left and right pans of a balance.

The fact that that exclusion of reflexivity is based on deliberations about the operation of measurement of weight on a double-pan scale indicates that (6) also has a *metrological* aspect, in the sense that it gives a prescription for the measurement of mass of bodies based on the measurement of their weight. If, on the basis of the operation on a double-pan scale described above, we find out that $Wx = Wy$, then we state that $mx = my$. Thus, while $Wx = Wy$ has the status of a *mesurans*, $mx = my$ has the status of a *mesurandum*, while $Bx = By$ delineates the type of entities to which both the *mesurans* and the *mesurandum* can be applied.

Given the structure of definition (6), it is reasonable to ask about the nature of the relation between the *definiens* and the *definiendum* in this definition. Seemingly, mass should be viewed as the ground in which the relation $=_{\mathrm{op}}$ has its basis — that is, its physical foundation. However, (6) does not state what mass *qua* mass is. It states only that mass is the entity shared by bodies x and y in the same amount; that is, the size of the magnitude *mass* as displayed by body x is the same as the size of the magnitude *mass* as displayed by body y, where this sameness of size was defined on the basis of the sameness of the size of Wx and the size Wy. And it holds also that the magnitude *weight* as a pre-dynamical magnitude cannot be the physical foundation of the magnitude *mass*.

How is the magnitude *mass* understood by Newton in the *Principia* — that is, in classical mechanics? Seemingly, it is possible to ground the magnitude *mass* itself in magnitude *weight*, the latter being now understood as a type of force to which a body with mass m is subjected on Earth and due to the action of which the body acquires the acceleration g. The relation, which is thus of dynamic nature and by means of which the magnitude m could then be defined is, in modern notation, $m =_{\mathrm{df}} W/g$.

However, for Newton's sequence of derivations of magnitudes it holds that the magnitude *weight*, understood now as a kind of force — that is, as a *dynamic* magnitude — cannot be defined independently from the magnitude *mass*. In terms of magnitudes, $W =_{\mathrm{df}} m{\cdot}g$. That this is so is readily seen from the following sequence of definitions in the *Principia* (1999: 403, 407):

> DEFINITION 2. *Quantity of motion is a measure of motion that arises from the velocity and the quantity of matter jointly.*
>
> DEFINITION 5. *Centripetal force is the force by which bodies are drawn from all sides, are impelled or in any way tend, toward some point as to a center.*
>
> DEFINITION 8. *The motive quantity of centripetal force is the measure of this force that is proportional to the motion it generates in a given time.*

In the commentary on the last definition Newton then declares: "An example is weight" (1999: 407) — that is, he subsumes weight under the category of centripetal force.

Thus, what is missing in classical mechanics, once the move from $Wx = Wy$ to $mx = my$ is performed by definition (6), is the next step — namely, the definition of the very magnitude *mass* that would be independent of its definition in (6). Once this definition has been given, in the next step, magnitude m defined in this way could be related to magnitude W understood in a dynamic way in a way different from $W =_{\mathrm{df}} m{\cdot}g$. Once both the new definition of m and the new relation between W and m have been given, what should follow is the introduction of the ratio $Wx = Wy$ and the ratio $mx = my$ and, finally, the introduction of the relation between these ratios in a way that differs from that given in (6).

To put it another way, what classical mechanics lacks is a definition of magnitude *mass* that would be independent of the magnitude *weight* understood in a pre-dynamical way and that could serve as the *mediating link* between, on the one hand, the move from $Wx =_{op} Wy$ to $mx = my$ as performed in (6) and, on the other hand, the move that would ultimately lead to a rederivation of the relation between $mx = my$ and $Wx = Wy$. In such a rederivation weight would be understood already in a way different from that given in (6) — namely, as a dynamical magnitude. In this sense it can be said that classical mechanics lacks a definition of magnitude *mass* that could serve as a *turning point* enabling such a rederivation.

In order to remedy this deficiency in classical mechanics, an alternative strategy to the application of the MDA can be suggested. With respect to (6) it is possible to identify the mass of body *x* with a *class* of *y*s for which $Wx =_{op} Wy$ holds. Such a strategy of elimination of the abstract entity was indicated in general terms by Peano: "Various authors have observed that one can always give the nominal definition of φx; just by putting φx = class of *y*s such that *xRy*" (1915: 111), where *xRy* corresponds to $p_{u,v}$ in the notation of Def.[13]

This alternative method thus stands for an elimination of the abstract entity in favor of a *class* of entities which stand in relation *p* to each other. This method, as I will now show, was suggested by Reichenbach.

4. REICHENBACH'S AND DUBISLAV'S CRITIQUE AND A REALISTIC ALTERNATIVE

Hans Reichenbach uses the term "class" in his interpretation of the employment of the MDA. He uses the following example of such an employment (1947, § 37: 209):

> The property of redness can be defined by abstracting it from a group of red objects. . . . First we must define the group of objects from which the abstraction is to be made. We then find that all these objects are connected by a symmetrical relation: in our example, the relation *of color-similarity*. Starting from a particular object, say, a certain rose, we thus define the class of objects which are color-similar to this rose. Instead of speaking of the common property we then simply speak of the class so defined; in other words, instead of saying that an object is red, we shall say that an object is a member of the class of things color-similar to this rose.

[13] One could say that here an extensional rather than an intensional definition is used; this point was suggested to me by one of the reviewers.

The general strategy he embarks upon in this example is as follows (1947, § 37: 209-210):

> The notion of property thus can be replaced by the notion of class; the abstraction of the common feature of the group of objects is replaced by the transition to the totality of the group. It is clear that every interconnective and symmetrical relation will give rise to a class. The interpretation of properties so obtained is very satisfactory because it eliminates unncessary entities, in Occam's sense.

Walter Dubislav (1929: 14) also claims that MDA violates Occam's rule "entia non sunt multiplicanda sine necessitate."[14] He views this method as based on a metaphysical axiom — that is, in fact, on a prejudice employed by its proponets in order to introduce ideal objects, using Weyl's terminology from his (1949). This axiom, in Dubislav's wording, is as follows (1929: 20):

> If in the framework of a discipline between its objects there exists a symmetric and transitive relation *R*, then there always exists in the form of an "ideal object" (that turns out not to be merely an expedient new formulation that only replaces an already known one, because then there would be no dispute!) a sufficiently determined common property of the objects in question.

From the point of view of my paper, the following critique by Dubislav of Peano's function φu in Def. is worth quoting. This function, in Dubislav's terminology "logical function" (1929: 21-22),

> suffers ... from an embarrassing indeterminacy that makes its employment comparatively speaking worthless. Viz., if one wants to get along in the further construction of the respective discipline without additional metaphysical presuppositions regarding the nature of the logical function in question, then it is not possible to correctly ground further statements about this function in any other way than in a round-about way by means of the relation *R* that is connected with it.

Both Reichenbach's strategy and Dubislav's critique are correct insofar as they identify the following weak spot in the MDA. This method serves the purpose of introducing an *abstractum* (Peano's φ), and the fact is that this *abstractum* is conceived by the method as Janus-faced. On the one hand, it is the result of reasoning leading from the knowledge of a type of relations jointly shared by certain entities (Peano's $p_{u,v}$) to the identification of a property as an *abstractum* they have in common. That knowledge has the status of a *reason* that justifies the introduction of this *abstractum* as *the reasoned*, *the justified*. On the other hand, in this method the implicit suggestion is present that the *abstractum* should have the status of a *real ground* that grounds the relation between those entities. However due to its structure —

[14] One could, of course, object against such an employment of Occam's rule by claiming that it should be applied only to fundamental objects.

and that is its limitation — the MDA can neither say what that *abstractum* is independently of that relation between entities nor say what its status is as a ground that grounds that relation between those entities.

Due to this limitation, Reichenbach suggests giving up the *abstractum* in favor of the class of entities, while Dubislav opts for an even more radical alternative — namely, to completely discard MDA as metaphysically burdened. Despite this difference between their respective approaches to the MDA, they share the view that this method represents the *ultima ratio* in trying to grasp the nature of that *abstractum*.

In my view, however, the MDA need not, and — in certain empirical sciences — in fact does not represent the *ultima ratio* method either for grasping the very *abstractum* or for grasping this *abstractum* as the ground that grounds the relation between entities, which in that method was the starting point. It is my contention that the MDA stands for an initial level, wherein the knowledge about the *abstractum* is produced for the first time and thus this level *cannot be bypassed*.

From this first level a second level can develop (and in certain empirical sciences it does develop), where the knowledge about the *abstractum* is produced *independently* of the knowledge about that relation between entities. Finally, once this second level is given, another level can develop (and in certain empirical sciences it does), where the knowledge is produced that discloses how that *abstractum* as a ground grounds that relation between those entities. Accordingly, the knowledge about the *abstractum* from the second level has the status of a mediating link between these two levels and represents a point of reversal — as described above for the case of the magnitude *mass* as missing in classical mechanics — from which then, finally, the "trip back" could start.

The second and third levels can thus be viewed as enabling what, as Dubislav correctly noticed, the MDA cannot deliver — namely, "a further construction of the respective discipline" (1929: 21), while at the same time we can regard these two levels jointly as a refutation of his claim that "it is not possible to correctly ground further statements about this function [φ] in any other way than by means of a round-about way by means of the relation R that is connected with it" (1929: 22).

As I will show, in David Ricardo's theory, the magnitude *labor*, initially introduced in his *Principles* by means of the method of definition by abstraction, acquired the status of that mediating link and point of reversal enabling such a "trip back." Accordingly, the production of knowledge from all the three levels can be discerned in that theory.

5. DAVID RICARDO ON THE MAGNITUDE *LABOR*

Ricardo starts the chapter "On Value" in the third edition of the *Principles* with the following statement: "The value of a commodity, or the quantity of any other commodity for which it will exchange, depends on the relative quantity of labour which is necessary for its production" (I: 11).[15] Ricardo then explains that, for him, the term "commodity" refers to products that not only are the result of the employment of a certain amount (Ricardo's "quantity") of labor but that have to fulfill two additional conditions.

First, "they may be multiplied, not in one country alone, but in many, almost without any assignable limit, if we are disposed to bestow the labour necessary to obtain them" (I: 11). Thus, Ricardo presupposes that commodities are products that are both *reproducible* and reproducible by *anyone* who can employ the necessary amount of labor and the necessary technology. Second, in order to be commodities, products have to be produced under conditions in which "competition operates without constraints" (I: 12).

Ricardo's linkage of perfect competition with reproducibility is related to the fact that in addition to the concept of *value* of a commodity he also employs the concept of *exchangeable value* (in the sense of the amount of a commodity that may be exchanged for an amount of some other commodity).

This can be readily seen at the very beginning of section 1 of the chapter "On Value," in which Ricardo employs the term "value of a commodity" in the sense of exchange ratios into which a quantity of a kind of commodity enters with respect to quantities of other kinds of commodities. He characterizes this type of value in the *Principles* also as "exchangeable value" (I: 13), "relative value" (I: 191), "comparative value" (I: 373), and "proportional value" (IV: 398). He then relates this type of value of a commodity to the amount of labor expended in its production.

Accordingly, I draw the conclusion that Ricardo's *initial point of departure* is the exchange ratios of commodities, based on which he, *then*, *determines* the ratios of amounts of labor expended on the production of these commodities. The reason why Ricardo imposes the conditions of reproducibility and free competition on the commodities that enter into exchange becomes clear when we realize that only under such an idealization can one escape extreme market situations in which the demand for a commodity massively exceeds its supply, in turn causing a disruption of its exchange ratios against other commodities.

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[15] In references to Ricardo's works I state, first, in Roman numerals, the volume from his *Works and Correspondence* (1951-1973) and then the page number.

Ricardo's inferences may be reconstructed, drawing on the notation of the MDA, as follows:

(7) $$Cx = Cy \rightarrow [(Lx = Ly) =_{\mathrm{df}} (EVx =_{\mathrm{op}} EVy)]$$

Here Cx and Cy stand for commodities x and y, respectively, being freely reproducible under the conditions of free competition; $Cx = Cy$ corresponds to hypothesis $h_{u,v}$ (in Peano's notation) — it picks out pairs to which the definition could be applied. All such pairs should share the property of being a commodity. "EV" stands for "exchangeable value" and "$EVx =_{\mathrm{op}} EVy$" expresses the ratio in which x and y are practically exchanged for each other, say "1 beaver = 2 deer," while "$Lx = Ly$" expresses the identity of the amounts of labor expended on x and y.

When comparing (7) with Peano's Def., the following difference between them becomes apparent. While Peano imposes on the relation $p_{u,v}$ the requirements of reflexivity, symmetry, and transitivity, $=_{\mathrm{op}}$ in (7) fulfills — as in the case of the practical weighing of bodies on a balance — only the last two requirements. No commodity can enter into an exchange relation with itself; only different commodities can enter into such a relation.

Like in the case of the definition of magnitude *mass* in (6), the definition of the magnitude *labor* in (7) has a metrological aspect. Once one finds out that a change has taken place in the ratio in which the amount of commodity x is exchanged for an amount of commodity y (i.e., in the ratio of their exchangeable or relative values), then, on the basis of (7), one can infer the amount of change in the ratio of labor in x to labor in y. This metrological aspect is readily seen in Ricardo's reflections about a hypothetical commodity always produced with an invariable amount of labor. In the first and second editions of the *Principles* Ricardo states the following concerning such a commodity (I: 17):

> If any one commodity could be found, which now and at all times required precisely the same quantity of labour to produce it, that commodity would be . . . eminently useful as a standard by which the variations of other things might be measured. . . . It is . . . of considerable use towards attaining a correct theory, to ascertain what the essential qualities of a standard are, that we may know the causes of the variation in the relative value of commodities.

In addition to the term "exchangeable value" and its synonyms, Ricardo employs the term "value" as synonymous with the terms "absolute value" (I: 21), "real value" (I: 191; II: 32; IX: 38), and "positive value" (IX: 2). They all pertain not to the exchange ratios of commodities but to their production and depend on the labor expended on them in this production. He expresses this dependence as follows: "All commodities having value are the result either of immediate labour, or of immediate and accumulated labour united" (IV: 379), where the latter stands for past labor embodied, for example, in machinery and buildings (I: 25-26) employed in the production by immediate labor.

What Ricardo thus performs is the move from $Lx = Ly$ as defined in (7) to magnitude *labor* as such, understood as that embodied in any commodity in the course of its production. Figure 1 expresses this move.

$$(Lx = Ly) \longrightarrow L$$

Figure 1. Ricardo's inference from the identity $Lx = Ly$ of the amounts of labor expended on commodities x and y to the magnitude *labor* as such

This labor as embodied in various amounts in different commodities then, in turn, determines the ratios in which they are exchanged — that is, their respective exchangeable values (I: 25). Thus, in addition to the level where Ricardo moves from the exchange ratios between commodities to the amounts of labor embodied in them, one can also discern a level where the movement is already going in the opposite direction — namely, from amounts of labor embodied in commodities to their exchange ratios.

The basis of this movement is the concept of labor which, according to Ricardo, is "the real foundation of exchangeable value" (I: 25). That this is really so can also be seen in his statement:

> Labour [is] the foundation of the value of commodities, and the comparative quantity of labour which is necessary to their production, the rule which determines the respective quantities of goods which shall be given in exchange for each other. (I: 87)

So, (7) stands for what was above viewed as the first level of knowledge production based on the employment of the MDA, and Figure 1 expresses what was understood as the second level. The third and ultimate level, with the sequences of magnitudes given by Ricardo, can be schematically expressed as in Figure 2:

$$\begin{tikzcd}
 & Lx \arrow[dr] & \\
L \arrow[ur] \arrow[dr] & & (Lx = Ly)\ {}_{\mathrm{df}}\!= (EVx = EVy) \\
 & Ly \arrow[ur] &
\end{tikzcd}$$

Figure 2. Ricardo's sequence from the magnitude *labor* to the magnitude *exchangeable value*

Here "L" stands for labor as such, and "Lx" and "Ly" stand for the amount of labor embodied in x and y, respectively. The arrows indicate the direction of Ricardo's reasoning leading to the derivation of $EVx = EVy$. This derivation has the status of a *rederivation* of the equality between *EVx* and *EVy* initially given in (7).

All three levels in their sequence and interconnections are unified in Figure 3.

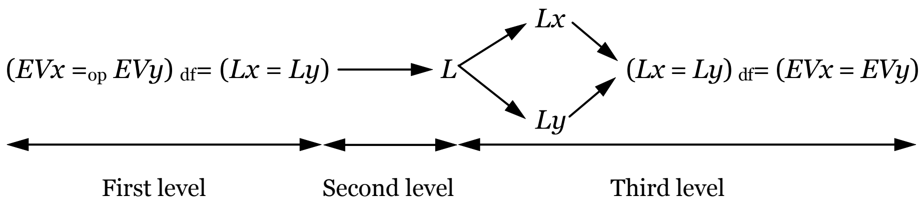

Figure 3. Three levels of production of knowledge about magnitudes assigned to commodities by Ricardo

The first level — production of knowledge about magnitudes based on the MDA — is the starting point from which the second level evolves and from which, in turn, the third level evolves. The second level, which functions as the mediating link between the first and third levels and as the point of “return” from $EVx =_{op} EVy$ back to $EVx = EVy$, is the knowledge about the very magnitude L — that is, the knowledge about labor as such given at this second level.

This knowledge is the basis for the difference between the *epistemological* status of $EVx =_{op} EVy$ and that of $EVx = EVy$, where this difference stands for an *extension of knowledge*. While $EVx =_{op} EVy$ expresses a knowledge about an equality that was discovered in the practical exchange of commodities, $EVx = EVy$ already expresses knowledge that is founded, first, on the knowledge that the exchanged commodities share a common ground — namely, that they are the product of labor as such expended on their production. It is founded, second, on the knowledge that the ratio in which the commodities are exchanged has its basis in the ratio of labor expended for production of each these commodities.

CONCLUSION: SEMANTICS FOR THE METHOD OF DEFINITION BY ABSTRACTION IN EMPIRICAL SCIENCES

The aim of this paper has been to analyze the structure of a specific type of definition by abstraction — namely, one that involves magnitudes of empirical sciences, both those which are already known as well as new ones, which are defined by means of this method on the basis of those already known. However, I was silent about three issues: the semantic characteristics of a definition, the semantic nature of a magnitude and, most importantly, the semantic characteristics of a definition by abstraction that involves magnitudes in its *definiens* and *definiendum*.

The necessity and importance of finding answers to these semantic issues was indicated by one of Mario Bunge's objections against Ernst Mach's proposal of an alternative definition to Newton's definition of the magnitude *mass* in the *Principia*.[16] According to Bunge, Mach committed the following *logical* mistake by viewing this statement as a definition (1992: 253-254):

> The logical mistake was to confuse equality with identity and in particular with definition. In fact, however, Mach's pseudo-postulate, like most physical laws, established an equality between two expressions which differ in meaning and therefore cannot be regarded as two sides of a definition. Indeed while "m_1/m_2" means "the inertia of body **1** relative to the inertia of body **2**," the symbol "$-a_2/a_1$" stands for a purely kinematic quantity. The equality is numerical not logical: it does not authorize us to eliminate one of the sides in favor of the other. Similarly, it is mistaken to regard "$f = ma$" as a definition of force in terms of m and a. It is not just a question of calling ma by the name f or conversely: the two concepts happen to be related in that way in classical mechanics.

Two points are readily seen in Bunge's critique. First, in Bunge's understanding of a definition, its *definiens* and *definiendum* should have the same meaning and, second, Bunge characterizes magnitudes of physics as concepts.

Let me, first, comment on how Bunge understands definitions.[17] This understanding is based on the widely accepted thesis that the *definiens* and the *definiendum* in a definition express the same meaning. In Frege's text "Logik in der Mathematik" this thesis is stated as follows (1913/1983: 224):

> Like the sentence, which generally is a composed sign, the thought that it expresses is composed too; and this in such a way that parts of the thought correspond to parts of the sentence. So, in general, also a group of signs that appears in a sentence will have a sense that is part of the thought. If now for such a group of signs a simple one . . . is introduced, then such a stipulation is a definition. The simple sign thereby acquires a sense — namely, the same that the group of sign has. . . . Through the introduction of the simple sign nothing is added from the point of view of content.

Frege described this sameness of content in "Über die Grundlegung der Geometrie" as follows:

> By defining, no knowledge comes about. . . . No definition extends knowledge, but is only a means to sum up a manifold content in a brief word or sign, and to make it in this way more manageable for us. (1903/1967: 263)

[16] Mach's statement, viewed by Bunge as a pseudo-postulate, is as follows: "Bodies of the same mass are called those which, when acting one upon another, confer on each other equal and opposite accelerations" (1901: 227); in symbols: $-a_2/a_1 = m_1/m_2$.

[17] Here I follow Daniela Glavaničová (2017).

Let me now turn to Bunge's characterization of magnitudes as concepts. From the point of view of intensional semantics, concepts are functions from possible worlds to a set of individuals as the concepts' extension.[18] Magnitudes can also be viewed as intensions — that is, functions defined on possible worlds. The specificity of magnitudes, compared to concepts, is that their extensions are not sets of individuals but sets of denominated numbers, for example, 5 kilograms, 3 inches, etc. The path from magnitudes as intensions to their extensions — that is, their size — is mediated by the employment of units of measurement.

In the case of the definitions (6) and (7), whose *definiens* and *definiendum* contain names of magnitudes, we face the following question: What is the relation between the intension expressed by their respective *definienda* and the intension expressed by their respective *definientia*?

To answer this question let me analyze the introduction, by means of the MDA, of magnitude *mass* in (6) and of magnitude *labor* in (7). Definition (6) is true in all possible worlds in which bodies exist; definition (7) is true in all possible worlds in which commodities exist. Outside its respective set of possible worlds neither of these definitions is applicable.[19]

Given the metrological nature, as delineated above, of definitions (6) and (7), their truth in all these possible worlds also means that, in all possible worlds wherein the respective entities (bodies, commodities) exist, the truth-values of the statement-form in the *definiendum* is the same as that of the statement-form in the *definiens*. Thus, in all these possible worlds the extension of the statement-form in the *definiendum* of (6) is the same as the extension of the statement-form in the *definiens* of (6); and the same holds for (7).

This identity of extensions that holds for definition (6) does not, of course, hold for definition $W =_{\mathrm{df}} m \cdot g$. The reason is readily seen from what was stated above about the difference between these definitions. According to the latter definition in all those possible worlds where two bodies have the same mass, these bodies can have different weights; according to definition (6), in all those possible worlds where two bodies have the same weight, they always have the same mass.

What conclusions relevant for a semantics of the MDA can then be drawn? The answer to this question depends on the type of semantic theory one chooses as a framework for this answer. Let me try to answer it in the framework of intensional semantics. As stated, for definition by abstraction it

[18] By intensional semantics I understand the semantics presented in (Carnap 1947).

[19] In the case of definition (7), this means that it cannot be applied to economies where goods are produced exclusively for subsistence and not for exchange; these goods are thus not commodities.

holds that the extension of its *definiendum* is the same as that of its *definiens* in all possible worlds wherein the respective type of entities exists. Thus, the conclusion — given the framework of intensional semantics — is that the intension (content) expressed by the *definiendum* is the same as that expressed by the *definiens*. Stated otherwise, the *definiens* and the *definiendum* are in this framework *meaning-equivalent*.

However, the content expressed by the *definiendum* in (6) is different from that expressed by its *definiens*. In the former we find the name of the dynamic magnitude *mass* while in the latter we find the name of a different magnitude — namely, *weight* — that has a pre-dynamic status. The same holds for (7), where the content expressed by its *definiendum* is different from the economic content of its *definiens*. The name of the magnitude *labor* appears in the former, while the name of a different magnitude — namely, *exchange value*, appears in the latter. Given these differences of contents, in both (6) and (7) the *definiendum* expresses, in comparison to the *definiens*, a surplus knowledge of magnitudes.

Two conclusions follow from my analysis thus far.

(1) The first conclusion is that a semantic theory for the MDA that would be able to unify all of the above characteristics of this method cannot be developed in the framework of intensional semantics. What could be a possible alternative to this framework? I will try to answer this question with the following line of reasoning.

Let us suppose that x and y are physical magnitudes embedded into the definition $y =_{\text{df}} e^x$ and valid for all possible worlds in which physical entities of a certain type exist, and let us replace e^x by its expansion into the series $1 + x + x^2/2! + x^3/3! + \ldots + x^k/k! + \ldots$.[20] We then have:

$$y =_{\text{df}} e^x$$

$$y =_{\text{df}} 1 + x + x^2/2! + x^3/3! + \ldots + x^k/k! + \ldots$$

However, if we choose the framework of intensional semantics, the interpretation of the semantics of these two definitions leads to a strange phenomenon. On the one hand, under the substitution into both expressions of the same measurement-values for the physical magnitude expressed as "x," in both definitions "y" refers to the same denominated numbers and thus the expressions "e^x" and "$1 + x + x^2/2! + x^3/3! + \ldots + x^k/k! + \ldots$" have the same extension in all those possible worlds in which physical entities of a certain type exist. From this, in the framework of intensional semantics, it follows that these expres-

[20] I owe the idea of replacing a function by its expansion into a series for the purpose of a semantical analysis to Hubert Schleichert (1966: 47-48).

sions have the same intension. On the other hand, the conclusion that "e^x" and "$1 + x + x^2/2! + x^3/3! + \ldots + x^k/k! + \ldots$" express the same intension seems counter-intuitive given the difference between the functions they express.

One possible way to escape this counter-intuitive sameness of intensions is to provide a different type of semantic clarification on the basis of *hyper-intensional* semantics. In the framework of such semantics, hyperintension would be an additional — with respect to intension — semantic entity, so that two different hyperintensions can lead to the same intension.[21] In such a way it would be possible to reconstruct the situation that obtains in the case of the MDA — namely, that while the *definiens* and *definiendum* express the same intension, the *definiendum* displays, when compared to the *definiens*, a surplus of knowledge.

What "e^x" and "$1 + x + x^2/2! + x^3/3! + \ldots + x^k/k! + \ldots$" express can then be viewed as two different hyperintensions, and since "y," under that substitution in both definitions, refers to the same extensions in all possible worlds, these two hyperintensions lead to the same intension; they are two different ways of identifying the same entity.

The lesson of my line of reasoning, then, is as follows. It is possible that, in that example of the substitution of the expansion of a function for this function, the phenomenon of hyperintension is at work and, even more importantly, we identify here two indicators for the presence of such a phenomenon. First, one faces — from the point of view of intensional semantics — meaning-equivalent expressions, and second, at the same time, this identity leads in the case of substitution to a situation that cannot be properly treated in the framework of this semantics.

These two indicators are present in the given characterization of the MDA. In a definition by abstraction the *definiens* and *definiendum* are, in the framework of intensional semantics, meaning-equivalent. At the same time, as the *definiendum* expresses a surplus of knowledge with regard to the *definiens*, one faces a counter-intuitive situation in the case of the substitution of the *definiens* for the *definiendum*. Since the *definiens* and *definiendum* differ from each other in terms of the knowledge they express, this substitution leads to a loss of the surplus knowledge obtained by the introduction of the new magnitude in the *definiendum*.

(2) The second conclusion is that a semantics that would be adequate to the MDA should be provided in the framework of a hyperintensional semantics wherein the thesis of identity of the meaning of the *definiens* and

[21] A candidate for such a hyperintension could be, for example, the entity labeled by Pavel Tichý as "construction." On this see Tichý (1988) and Duží, Jespersen, Materna (2010).

definiendum no longer holds.[22] The creation of such a semantics for the MDA in that framework is an urgent task that must be addressed in the near future.

APPENDIX: CIRCULAR NATURE OF NEWTON'S INTRODUCTION OF THE MAGNITUDE *MASS* IN THE *PRINCIPIA*

As stated in the body of the article, an alternative path of reasoning about the magnitude *mass* is suggested in Newton's commentary on Definition 1. Then a sequence of definitions follows, the most relevant for this article being the following (1999: 404, 407):

> DEFINITION 2. *Quantity of motion is a measure of motion that arises from the velocity and the quantity of matter jointly.*

> DEFINITION 3. *Inherent force of matter is the power of resisting by which every body, so far as it is able, perseveres in its state of resting or of moving uniformly straight forward.*

Then, in the commentary on Definition 3, Newton draws on the magnitude *mass* introduced in Definition 1 by stating: "This force is always proportional to the body" (1999: 407).

> DEFINITION 5. *Centripetal force is the force by which bodies are drawn from all sides, are impelled or in any way tend, toward some point as to a center.*

> DEFINITION 8. *The motive quantity of centripetal force is the measure of this force that is proportional to the motion it generates in a given time.*

In the commentary on the last of these definitions Newton states (1999: 407-408):

> An example is weight, which is greater in a larger body and less in a smaller body; and in one and the same body is greater near the earth and less out in the heavens. . . . motive [quantity of] force [arises] from accelerative force and quantity of matter jointly. As a consequence, near the surface of the earth, where . . . the force that produces gravity is the same in all bodies universally, the motive quantity, or weight, is as the body, but in an ascent to regions where the accelerative gravity becomes less, the weight will decrease proportionally and will always be as the body and the accelerative gravity jointly.

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[22] This precludes all those hyperintensional semantics that stick to the thesis of identity of the meaning of the *definiens* and that of the *definiendum* (e.g., Jago 2014: 79-82).

Newton's commentary on Definition 1 mentions his experiments with pendulums; these experiments are based on the physics given in Proposition 24 and four of its corollaries in Book II of the *Principia* (1999: 700-701):

> PROPOSITION 24. In simple pendulums whose centers of oscillation are equally distant from the center of suspension, the quantities of matter are in a ratio compounded of the weights and the squared ratio of the times of oscillation in a vacuum.
>
> Corollary 1. And thus if the times are equal, the quantities of matter in the bodies will be as their weights.
>
> Corollary 5. And universally, the quantity of matter in a bob of a simple pendulum is as the weight and the square of the time directly and the length of the pendulum inversely.
>
> Corollary 6. But in a nonresisting medium also, the quantity of matter in the bob of a simple pendulum is as the relative weight and the square of the time directly and the length of the pendulum inversely. For the relative weight is the motive force of a body in any heavy medium . . . and thus fulfills the same function in such a nonresisting medium as absolute weight in a vacuum.
>
> Corollary 7. And hence a method is apparent both for comparing bodies with one another with respect to the quantity of matter in each, and for comparing the weights of one and the same body in different places in order to find out the variation in its gravity. And by making experiments of the greatest possible accuracy, I have always found that the quantity of matter in individual bodies is proportional to the weight.

The experiments mentioned in the last of these corollaries are exactly those to which Newton refers in Definition 1 and they are described in the commentary on Proposition 6 of Book III as follows (1999: 807):

> I got two wooden boxes, round and equal. I filled one of them with wood and I suspended the same weight of gold . . . in the center of oscillation of the other. The boxes, hanging by equal eleven foot cords, made pendulums exactly like each other with respect to their weight, shape, and air resistance. Then, when placed close to each other [and set into vibration], they kept swinging back and forth together with equal oscillations for a long time.

Newton repeated these experiments with other substances like silver, lead, glass, sand, common salt, water, and wheat, and he obtained the same results — namely, that the masses of the bobs attached to these pendula are to each other as are to each other the motive forces acting on the substances and, thus, as their weights.

The reconstruction of that dependence means that, contrary to the declared introduction of the magnitude *mass* by means of the magnitudes *volume* and *density* in Definition 1, in Newton's sequence of introductions of physical magnitudes, the magnitude *mass* is in fact introduced independently from these two magnitudes.

What has been shown so far is that the magnitude *mass* is transferred from Proposition 24 (Book II) via Proposition 6 (Book III) into the section *Definitions*. But a closer look at the proof of the former proposition discloses that it is based on the prior knowledge of the magnitude labeled by Newton as "weight," understood as a motive force. The proof is as follows (1999: 701):

> [1.] For the velocity that a given force can generate in a given time in a given quantity of matter is as the force and the time directly and the matter inversely. The greater the force, or the greater the time, or the less the matter, the greater the velocity that will be generated. This is manifest from the second law.[23]
>
> [2.] Now if the pendulums are of the same length, the motive forces in the places equally distant from the perpendicular are as the weights; and thus if two oscillating bodies describe equal arcs and if the arcs are divided into equal parts, then, since the times in which the bodies describe single corresponding parts of the arcs are as the times of the whole oscillations, the velocities in corresponding parts of the oscillations will be to one another as the motive forces and the whole times of the oscillations directly and the quantities of matter inversely; and thus the quantities of matter will be as the forces and the times of oscillations directly and the velocities inversely.
>
> [3.] But the velocities are inversely as the times, and thus the times are directly, and the velocities are inversely as the square of the times, and therefore the quantities of matter are as the motive forces and the square of times, that is, as the weights and the square of the times. Q.E.D.

And where does the understanding of the magnitude *weight* as a motive force come from? It comes from the commentary on Definition 8, in which weight is viewed as an example of the "motive quantity of the centripetal force." This quantity is understood in Definition 8 as a measure computed on the basis of the time-variation of the magnitude *motion*. And where does, in turn, the understanding of the magnitude *motion* come from? It comes from Definition 2, in which it is defined by the multiplication of the magnitudes *mass* and *velocity*, the former magnitude going back to Definition 1.

Thus, Newton's reasoning about magnitude *mass* is circular. It goes from Definition 1, via Definitions 2 and 8, the Second Law, Proposition 24 (Book II), and Proposition 6 (Book III) back to Definition 1. Figure 4 expresses this circularity:[24]

[23] The second law is as follows: "A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed" (1999: 417).

[24] I have eliminated in this figure Definition 3 because it is not a necessary element for the represented circularity.

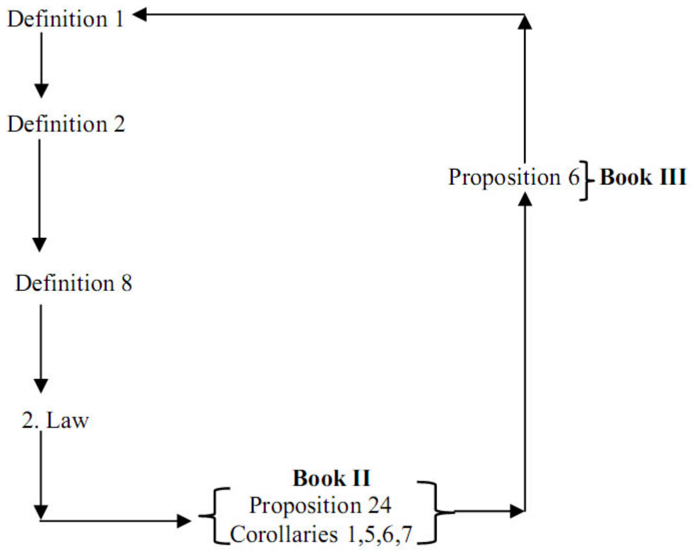

Figure 4. Circular nature of Newton's introduction of the magnitude *mass* in the *Principia*

The path from Definition 1 to Proposition 6 (Book III) at the same time discloses the way in which Newton attempted to construct his *overall* physical theory in the *Principia*. This overall theory involves two theories. One is a *general* theory, center-stage taken here by the sections *Definitions* and *Axioms or Laws of Motion* about the effects of *any* type of force (be it gravity, magnetic force, etc.) on bodies characterized by the magnitude *mass* as inertial mass. The second theory is, in comparison to that general theory, a *particular* theory about the effects of just one type force — namely, the effects of gravity on bodies characterized by the magnitude *mass* as gravitational mass.

The particular theory is based on the general theory. The point of connection between these theories is Definition 8, defining the motive quantity of centripetal force together with the comment on it:

> An example is weight, which is greater in a larger body and less in a smaller body; and in one and the same body is greater near the earth and less out in the heavens (Newton 1999: 407).[25]

Stated otherwise, the magnitudes and their relations in Newton's theory about effects of the force of gravity are not independent from the magnitudes

[25] This role of Definition 8 was brought to my attention by one the reviewers who stated that it is in fact a preparation of describing a unique feature of the gravitational force, which is proportional to mass.

and their relations in that general theory; without the former magnitudes and their relations, the latter magnitudes and their relations cannot be introduced.

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